Predictive Labeling with Dependency Pairs using SAT

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CADE
Bremen

Termination of Rewriting

Example

$$\begin{array}{rclrcl}
\mathsf{min}(x,0) & \rightarrow & 0 & \mathsf{max}(x,0) & \rightarrow & x \\
\mathsf{min}(0,y) & \rightarrow & 0 & \mathsf{max}(0,y) & \rightarrow & y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) & \rightarrow & \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) & \rightarrow & \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) & \rightarrow & \mathsf{s}(x) & x - 0 & \rightarrow & x \\
\mathsf{gcd}(\mathsf{s}(x),0) & \rightarrow & \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) & \rightarrow & x - y
\end{array}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \rightarrow \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Termination of Rewriting

Example

$$
\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x-0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) &\to x-y
\end{aligned}
$$

$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to$ gcd(max(x,y) - min(x,y), s(min(x,y)))

$\mathsf{gcd}(6,4)$

Termination of Rewriting

Example

$\mathsf{min}(x,0) \to 0$	$\mathsf{max}(x,0) \to x$
$\mathsf{min}(0,y) \to 0$	$\mathsf{max}(0,y) \to y$
$\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{min}(x,y))$	$\mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{max}(x,y))$
$\mathsf{gcd}(0,\mathsf{s}(x)) \to \mathsf{s}(x)$	$x - 0 \to x$
$\mathsf{gcd}(\mathsf{s}(x),0) \to \mathsf{s}(x)$	$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6,4) \to \mathsf{gcd}(\mathsf{max}(5,3) - \mathsf{min}(5,3), \mathsf{s}(\mathsf{min}(5,3)))$

Termination of Rewriting

Example

$$\begin{aligned}
\mathrm{min}(x,0) &\rightarrow 0 & \mathrm{max}(x,0) &\rightarrow x \\
\mathrm{min}(0,y) &\rightarrow 0 & \mathrm{max}(0,y) &\rightarrow y \\
\mathrm{min}(\mathrm{s}(x),\mathrm{s}(y)) &\rightarrow \mathrm{s}(\mathrm{min}(x,y)) & \mathrm{max}(\mathrm{s}(x),\mathrm{s}(y)) &\rightarrow \mathrm{s}(\mathrm{max}(x,y)) \\
\mathrm{gcd}(0,\mathrm{s}(x)) &\rightarrow \mathrm{s}(x) & x - 0 &\rightarrow x \\
\mathrm{gcd}(\mathrm{s}(x),0) &\rightarrow \mathrm{s}(x) & \mathrm{s}(x) - \mathrm{s}(y) &\rightarrow x - y \\
\mathrm{gcd}(\mathrm{s}(x),\mathrm{s}(y)) &\rightarrow \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$\mathrm{gcd}(6,4) \rightarrow^{+} \mathrm{gcd}(\mathrm{s}(\mathrm{max}(4,2)) - \mathrm{min}(5,3), \mathrm{s}(\mathrm{min}(5,3)))$

Termination of Rewriting

Example

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathsf{min}(x,0) & \to & 0 & \mathsf{max}(x,0) & \to & x \\
\mathsf{min}(0,y) & \to & 0 & \mathsf{max}(0,y) & \to & y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) & \to & \mathsf{s}(x) & x-0 & \to & x \\
\mathsf{gcd}(\mathsf{s}(x),0) & \to & \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) & \to & x-y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \multicolumn{4}{l}{\mathsf{gcd}(\mathsf{max(x,y)}\ \text{-}\ \mathsf{min(x,y)},\ \mathsf{s(min(x,y))})}
\end{array}$$

$$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{max}(3,1))) - \mathsf{min}(5,3), \mathsf{s}(\mathsf{min}(5,3)))$$

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x\\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y\\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y))\\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x-0 &\to x\\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) &\to x-y\\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{max}(2,0)))) - \mathsf{min}(5,3), \mathsf{s}(\mathsf{min}(5,3)))$

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x\\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y\\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y))\\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x\\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y\\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{s}(2))) - \mathsf{min}(5,3), \mathsf{s}(\mathsf{min}(5,3)))$

Termination of Rewriting

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ \mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))} \end{aligned}$$

$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(5 - 3, 4)$

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
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\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y)))
\end{aligned}$$

$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(2,4)$

Termination of Rewriting

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(2,2)$

Termination of Rewriting

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x-0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) &\to x-y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(0,2)$

Termination of Rewriting

Example

$$
\begin{array}{rclcrcl}
\mathsf{min}(x,0) & \to & 0 & & \mathsf{max}(x,0) & \to & x \\
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\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{min}(x,y)) & & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) & \to & \mathsf{s}(x) & & x - 0 & \to & x \\
\mathsf{gcd}(\mathsf{s}(x),0) & \to & \mathsf{s}(x) & & \mathsf{s}(x) - \mathsf{s}(y) & \to & x - y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \multicolumn{5}{l}{\text{gcd(max(x,y) - min(x,y), s(min(x,y)))}}
\end{array}
$$

$\mathsf{gcd}(6,4) \to^{+} 2$

Definition

A TRS is terminating if it does not admit an infinite rewrite sequence.

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Termination methods

Knuth-Bendix order, polynomial interpretations, lexicographic path order, multiset order, multiset path order, recursive path order, semantic path order, recursive decomposition order, transformation order, elementary interpretations, well-founded monotone algebra, general path order, semantic labeling, type introduction, freezing, top-down labeling, dependency pair method, matchbounds, size-change principle, predictive labeling, uncurrying, matrix interpretations, quasi-periodic interpretations, bounded increase . . .

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AProVE, Cariboo, Cime, JamBox, MatchBox, MultumNonMulta, MuTerm, Teparla, Torpa, TPA, TTT, TTTbox, . . .

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Outline

1. Theory
   - Semantic Labeling (SL)
   - Predictive Labeling (PL)
   - Dependency Pairs (DP)

2. Practice
   - SAT encoding
   - Experimental results

3. Conclusions

Outline

1. Theory
    - Semantic Labeling (SL)
    - Predictive Labeling (PL)
    - Dependency Pairs (DP)

2. Practice
    - SAT encoding
    - Experimental results

3. Conclusions

Semantic Labeling (SL)

Example

Is this TRS terminating?

$$
\begin{aligned}
\mathsf{f}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{f}(x,y))\\
\mathsf{f}(x,\mathsf{c}) &\rightarrow \mathsf{c}\\
\mathsf{f}(\mathsf{c},y) &\rightarrow \mathsf{c}\\
\mathsf{g}(x,\mathsf{c}) &\rightarrow x\\
\mathsf{g}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{g}(x,y))\\
\mathsf{g}(\mathsf{c},y) &\rightarrow y\\
\mathsf{h}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{h}(x,y)\\
\mathsf{h}(x,\mathsf{c}) &\rightarrow x\\
\mathsf{l}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{l}(\mathsf{h}(\mathsf{g}(x,y),\mathsf{f}(x,y)),\mathsf{s}(\mathsf{f}(x,y)))
\end{aligned}
$$

Semantic Labeling (SL)

Example

How about this one?

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}
$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 && x \geq 0 \\
\mathsf{min}(0, y) &\to 0 && 0 \geq 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) && 2x + 1 \geq 2x + 1 \\
\mathsf{max}(x, 0) &\to x && x \geq x \\
\mathsf{max}(0, y) &\to y && y \geq y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) && 2x + 2y + 2 > 2x + 2y + 1 \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y && 2x + 1 > x \\
x - 0 &\to x && x \geq x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) && 0 \geq 0
\end{aligned}
$$

$$
0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y
$$

$$
-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0
$$

Semantic Labeling (SL)

Example

$$\begin{aligned}
\mathsf{min}(x,0) &\to 0\\
\mathsf{min}(0,y) &\to 0\\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y))\\
\mathsf{max}(x,0) &\to x\\
\mathsf{max}(0,y) &\to y\\
\mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y))\\
\mathsf{s}(x)-\mathsf{s}(y) &\to x-y\\
x-0 &\to x \qquad i,j\in\mathbb{N}\\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y)))\\
\mathsf{gcd}_i(x,y) &\to \mathsf{gcd}_j(x,y) \qquad i>j
\end{aligned}$$

$$0_{\mathbb{N}}=0 \quad \mathsf{s}_{\mathbb{N}}(x)=2x+1 \quad \mathsf{min}_{\mathbb{N}}(x,y)=x \quad \mathsf{max}_{\mathbb{N}}(x,y)=x+y$$

$$-_{\mathbb{N}}(x,y)=x \quad \mathsf{gcd}_{\mathbb{N}}(x,y)=0$$

$$\ell_{\mathsf{gcd}}(x,y)=2x+y$$

Semantic Labeling (SL)

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}$$

⇒ LPO applicable

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathsf{gcd}}(x, y) = 2x + y$$

Semantic Labeling (SL)

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}$$

$\Rightarrow$ LPO applicable

$\ldots > \mathsf{gcd}_1 > \mathsf{gcd}_0 > \{\mathsf{min}, \mathsf{max}, -\} > \mathsf{s}$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathsf{gcd}}(x, y) = 2x + y$$

Semantic Labeling (SL)

Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$

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  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$
- Labeling:
  - for every $f \in \mathcal{F}$ a set of labels $L_f \subseteq A$
  - labeling functions $\ell_f : A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$

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- Labeling of terms (for a variable assignment $\alpha : \mathcal{V} \rightarrow A$).

$$\mathrm{lab}_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathrm{lab}_\alpha(t_1), \ldots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathrm{lab}_\alpha(t_1), \ldots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$

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with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$

- Labeled TRS:
  - $\mathcal{R}_{\mathsf{lab}} = \{\mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r) \mid l \to r \in \mathcal{R}, \alpha : \mathcal{V} \to A\}$
  - $\mathsf{Decr} = \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid f \in \mathcal{F}, L_f \neq \varnothing; a, b \in L_f, a > b\}$

Semantic Labeling (SL)

Theorem (Zantema, 1995)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a weakly-monotone $\mathcal{F}$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{R} \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Predictive Labeling (PL)

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \mathrm{gcd}(0,s(x)) &\to s(x) & x-0 &\to x \\ \mathrm{gcd}(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \mathrm{gcd}(s(x),s(y)) &\to \mathrm{gcd}(\text{max(x,y) - min(x,y), s(min(x,y))}) \end{aligned}$$

$$0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = 2x+1 \quad \min_{\mathbb{N}}(x,y) = x \quad \max_{\mathbb{N}}(x,y) = x+y$$
$$-_{\mathbb{N}}(x,y) = x \quad \mathrm{gcd}_{\mathbb{N}}(x,y) = 0$$

Problem: how to obtain a quasi-model?

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\rightarrow 0 & \mathsf{max}(x, 0) &\rightarrow x \\
\mathsf{min}(0, y) &\rightarrow 0 & \mathsf{max}(0, y) &\rightarrow y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\rightarrow \mathsf{s}(x) & x - 0 &\rightarrow x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\rightarrow \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\rightarrow x - y \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}
$$

$$
0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y
$$

$$
-_{\mathbb{N}} (x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0
$$

Problem: how to obtain a quasi-model?

⇒ use predictive labeling (quasi-model constraints only for usable rules)

Predictive Labeling (PL)

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}(\text{max(x,y) - min(x,y), s(min(x,y))})$$

Computation of usable rules:

1. Look at symbols that will get labels.

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}
$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}(\text{max(x,y) - min(x,y)}, \text{s(min(x,y))})$$

Computation of usable rules:

1. Look at symbols that will get labels.
2. Look at their subterms.

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \gcd(\max(x,y) - \min(x,y), s(\min(x,y))) \end{aligned}$$

Computation of usable rules:

2. Look at their subterms.
3. All function symbols occurring there are usable.

Predictive Labeling (PL)

Example

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\mathrm{min}(x,0) &\to 0 & \mathrm{max}(x,0) &\to x \\
\mathrm{min}(0,y) &\to 0 & \mathrm{max}(0,y) &\to y \\
\mathrm{min}(\mathrm{s}(x),\mathrm{s}(y)) &\to \mathrm{s}(\mathrm{min}(x,y)) & \mathrm{max}(\mathrm{s}(x),\mathrm{s}(y)) &\to \mathrm{s}(\mathrm{max}(x,y)) \\
\mathrm{gcd}(0,\mathrm{s}(x)) &\to \mathrm{s}(x) & x - 0 &\to x \\
\mathrm{gcd}(\mathrm{s}(x),0) &\to \mathrm{s}(x) & \mathrm{s}(x) - \mathrm{s}(y) &\to x - y \\
\mathrm{gcd}(\mathrm{s}(x),\mathrm{s}(y)) &\to \mathrm{gcd}(\mathrm{max}(\mathrm{x,y}) \text{ - } \mathrm{min}(\mathrm{x,y}), \mathrm{s}(\mathrm{min}(\mathrm{x,y})))
\end{aligned}$$

Computation of usable rules:

3. All function symbols occurring there are usable.
4. Also the symbols that depend on them.

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x\\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y\\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y))\\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x\\
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Computation of usable rules:

4. Also the symbols that depend on them.
5. Semantics needed only for usable symbols.

Predictive Labeling (PL)

Example

$$\min(x, 0) \to 0 \qquad \max(x, 0) \to x$$

$$\min(0, y) \to 0 \qquad \max(0, y) \to y$$

$$\min(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\min(x, y)) \qquad \max(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\max(x, y))$$

$$\mathsf{gcd}(0, \mathsf{s}(x)) \to \mathsf{s}(x) \qquad x - 0 \to x$$

$$\mathsf{gcd}(\mathsf{s}(x), 0) \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{gcd}(\mathsf{max(x,y)\ -\ min(x,y),\ s(min(x,y))})$$

Computation of usable rules:

5. Semantics needed only for usable symbols.
6. Quasi-model constraints only required for usable rules.

Predictive Labeling (PL)

Example

$$\mathsf{min}(x,0) \to 0 \qquad \mathsf{max}(x,0) \to x$$

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$$\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{min}(x,y)) \qquad \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{max}(x,y))$$

$$\mathsf{gcd}(0,\mathsf{s}(x)) \to \mathsf{s}(x) \qquad x - 0 \to x$$

$$\mathsf{gcd}(\mathsf{s}(x),0) \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

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Computation of usable rules:

1. Look at symbols that will get labels.
2. Look at their subterms.
3. All function symbols occurring there are usable.
4. Also the symbols that depend on them.
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Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \unlhd r$

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- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \rightarrow r \in \mathcal{R}, g(\ldots) \unlhd r$
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- $$\mathcal{G}_\ell(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \mathcal{F}(t_1)^* \cup \cdots \cup \mathcal{F}(t_n)^* & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing, \\ \mathcal{G}_\ell(t_1) \cup \cdots \cup \mathcal{G}_\ell(t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \end{cases}$$

Predictive Labeling (PL)

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- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$

Predictive Labeling (PL)

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- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$
- $\mathcal{U}(\mathcal{R}, \ell) = \{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{G}_\ell(\mathcal{R})\}$

Predictive Labeling (PL)

Theorem (Hirokawa, Middeldorp, 2006)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a weakly-monotone $\sqcup$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Dependency Pairs (DP)

Example

$$\begin{aligned}
\mathsf{min}(x,0) &\rightarrow 0 & \mathsf{max}(x,0) &\rightarrow x \\
\mathsf{min}(0,y) &\rightarrow 0 & \mathsf{max}(0,y) &\rightarrow y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\rightarrow \mathsf{s}(x) & x - 0 &\rightarrow x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\rightarrow \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\rightarrow x - y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\rightarrow \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

Dependency Pairs (DP)

Example

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\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}
$$

⇒ apply dependency pairs

Dependency Pairs (DP)

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}(\mathsf{max}(\mathsf{x,y})\ \text{-}\ \mathsf{min}(\mathsf{x,y}),\ \mathsf{s}(\mathsf{min}(\mathsf{x,y})))$$

Dependency Pairs (DP)

Example

$$
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\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\rightarrow \mathsf{s}(x) & x - 0 &\rightarrow x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\rightarrow \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\rightarrow x - y
\end{aligned}
$$

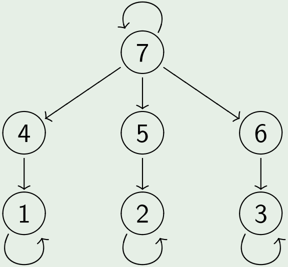
$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \rightarrow \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$$
\begin{aligned}
&(1) & \mathsf{minus}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{minus}^\sharp(x, y) & &(5) & \mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{min}^\sharp(x, y) \\
&(2) & \mathsf{min}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{min}^\sharp(x, y) & &(6) & \mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{max}^\sharp(x, y) \\
&(3) & \mathsf{max}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{max}^\sharp(x, y) \\
&(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{minus}^\sharp(\mathsf{max}(x, y), \mathsf{min}(x, y)) \\
&(7) & \mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{gcd}^\sharp(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}
$$

Dependency Pairs (DP)

Example

$$\begin{array}{llll} (1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) \\ (2) & \mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) & (6) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) \\ (3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) & & \\ (4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) & & \\ (7) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) & & \end{array}$$



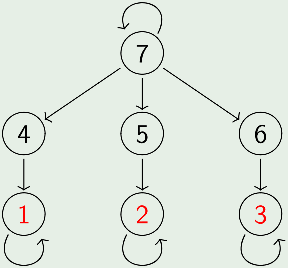
Dependency Pairs (DP)

Example

(1)	$\mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y)$	(5)	$\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)$
(2)	$\mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)$	(6)	$\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y)$
(3)	$\mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y)$		

(4) $\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y))$

(7) $\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y)))$



Dependency Pairs (DP)

Example

(1) $\mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y)$

(2) $\mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)$

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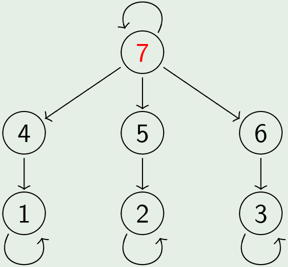

Proving termination with PL

$$\begin{tikzcd}
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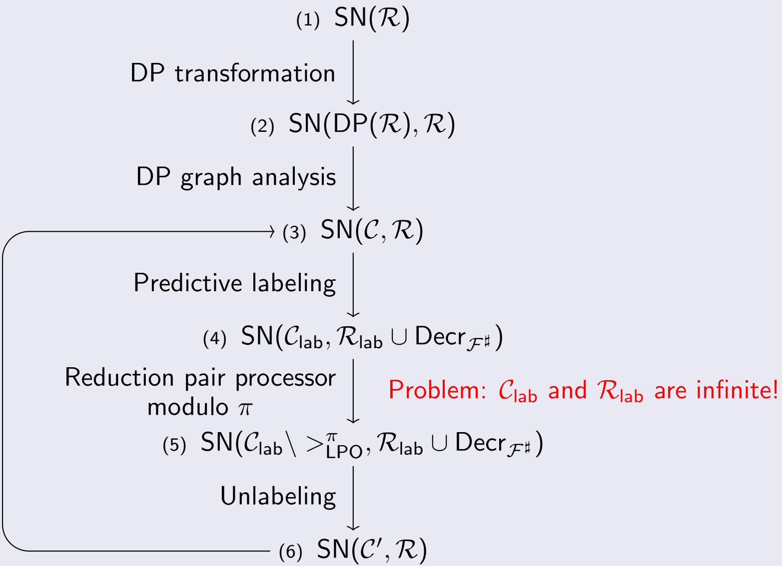

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Problem: $\mathcal{C}_{\mathsf{lab}}$ and $\mathcal{R}_{\mathsf{lab}}$ are infinite!

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PL in the DP setting

Theorem

A DP problem $(\mathcal{P}, \mathcal{R})$ *is finite if there exists:*

- *a weakly-monotone* $\sqcup$*-algebra and*
- *a weakly-monotone labeling* ℓ

such that:

- $\mathcal{P} \subseteq \mathsf{DP}(\mathcal{R})$
- $\mathcal{R}$ *is finitely branching,*
- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $(\mathcal{P}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr})$ *is finite.*

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1 Theory
- Semantic Labeling (SL)
- Predictive Labeling (PL)
- Dependency Pairs (DP)

2 Practice
- SAT encoding
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3 Conclusions

How to search for termination proofs using PL + DP?

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- choose which function symbols to label (L_f),

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⇒ Idea: use SAT

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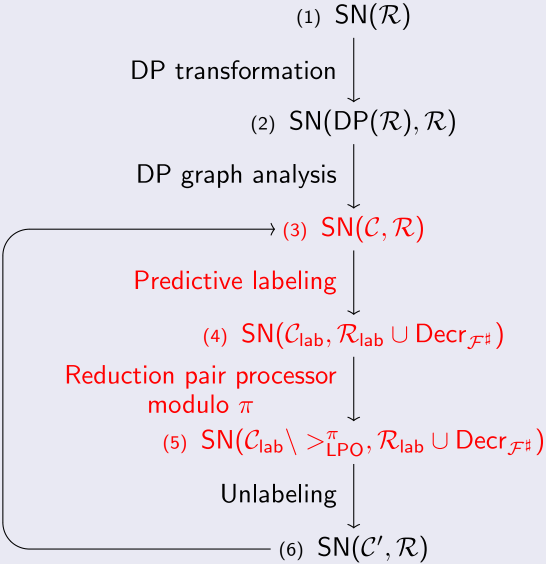

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Quasi-model constraints

- Symbol dependencies:

$$\Delta_f(\mathcal{P},\mathcal{R}) = \bigcup_{l \to r \in \mathcal{P} \cup \mathcal{R}} \{g \in \mathcal{F}(t) \mid \mathrm{root}(t) = f \text{ and } t \trianglelefteq l \text{ or } t \trianglelefteq r\}.$$

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- SAT encoding:

$$\ulcorner f_i \succ g_j \urcorner \;=\; \ulcorner f_{\mathrm{L}} >_{\mathbb{N}} g_{\mathrm{L}} \urcorner \vee \Big(\ulcorner f_{\mathrm{L}} =_{\mathbb{N}} g_{\mathrm{L}} \urcorner \wedge \mathsf{L}_f \wedge \mathsf{L}_g \wedge \big(\ulcorner i >_{\mathcal{A}} j \urcorner \vee \\ (\ulcorner i \gtrsim_{\mathcal{A}} j \urcorner \wedge \ulcorner f_{\mathrm{SL}} >_{\mathbb{N}} g_{\mathrm{SL}} \urcorner)\big)\Big)$$

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Experimental results

Table: Experiments with TPA on 864 TRSs from the TPDB version 3.2.

		60 seconds timeout			10 minutes timeout		
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R. Thiemann, A .Middeldorp.
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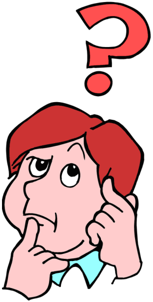
The end



Thank you for your attention.