Arctic Termination ... Below Zero

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Outline

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1. Introduction
2. Monotone Algebras
3. Polynomial and Matrix Interpretations
4. Arctic Interpretations
5. Arctic Below Zero Interpretations
6. Certification
7. Evaluation
8. Conclusions

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$\Longrightarrow$ This talk concerns a new method for proving termination, its automation and certification.

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Basic definitions

Given TRSs $\mathcal{R}$ and $\mathcal{S}$ define:

- *top rewrite relation:* $t \overset{\text{top}}{\to}_{\mathcal{R}} u$ if and only if there is a rewrite rule $\ell \to r \in \mathcal{R}$ and a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\Sigma, \mathcal{V})$ such that $t = \ell\sigma$ and $u = r\sigma$.

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- *relative top termination:* $\mathrm{SN}(\overset{\text{top}}{\to}_{\mathcal{R}} / \to_{\mathcal{S}})$ (important in the dependency pairs setting).

Monotone algebras

Definition (Monotonicity)

An operation $[f] : A \times \cdots \times A \to A$ is *monotone* with respect to a binary relation $\rhd$ on A if
$a_i \rhd a'_i \implies [f](a_1, \ldots, a_i, \ldots a_n) \rhd [f](a_1, \ldots, a'_i, \ldots, a_n)$.

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Definition (Monotone Σ-algebras)

A *weakly monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a Σ-algebra $(A, [\cdot])$ equipped with two binary relations $>$, $\gtrsim$ on A such that

- $>$ is well-founded;
- $> \cdot \gtrsim \subseteq >$;
- for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $\gtrsim$.

An *extended monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a weakly monotone Σ-algebra $(A, [\cdot], >, \gtrsim)$ in which moreover for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $>$.

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Theorem

Let $\mathcal{R}, \mathcal{R}', \mathcal{S}, \mathcal{S}'$ *be TRSs over a signature* Σ*,* $(A, [\cdot], >, \gtrsim)$ *be an extended monotone* Σ*-algebra such that:*

- $\forall_\alpha\ [\ell]_\alpha \gtrsim [r]_\alpha$ *for every rule* $\ell \to r$ *in* $\mathcal{R} \cup \mathcal{S}$ *and*
- $\forall_\alpha\ [\ell]_\alpha > [r]_\alpha$ *for every rule* $\ell \to r$ *in* $\mathcal{R}' \cup \mathcal{S}'$

Then $\mathrm{SN}(\to_{\mathcal{R}} / \to_{\mathcal{S}})$ *implies* $\mathrm{SN}(\to_{\mathcal{R}} \cup \to_{\mathcal{R}'} \ / \ \to_{\mathcal{S}} \cup \to_{\mathcal{S}'})$*.*

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$$\begin{aligned} [x * (y + z)] &= 2x + 2(y + z + 2) + 2x(y + z + 2) + 1 \\ [x * y + x * z] &= (2x + 2y + 2xy + 1) + (2x + 2z + 2xz + 1) + 2 \end{aligned}$$

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$$[x * (y + z)] = 2x + 2y + 2z + 4 + 2xy + 2xz + 4x + 1$$
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- To obtain strict monotonicity we require that for every interpretation $[f(x_1, \ldots, x_n)]$, $\forall_i \; \exists_{c>0} \; cx_i \in [f(x_1, \ldots, x_n)]$.

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- Now we need to restrict to linear interpretations.
- Strict monotonicity ensured if for every interpretation $[f(x_1, \ldots, x_n)] = F_1 x_1 + \ldots F_n x_n + \vec{f}$ we have $\forall_i \; (F_i)_{1,1} > 0$.

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- Problem: well-foundedness of $>$ (as $-\infty \gg -\infty$)

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Example arctic proof

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1. Introduction
2. Monotone Algebras
3. Polynomial and Matrix Interpretations
4. Arctic Interpretations
5. **Arctic Below Zero Interpretations**
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$$\begin{aligned} \mathrm{cond}(\mathrm{true}, x, y) &\to \mathrm{cond}(\mathrm{gr}(x, y), \mathrm{p}(x), y), & \mathrm{gr}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{gr}(x, y), \\ \mathrm{gr}(0, x) &\to \mathrm{false}, & \mathrm{gr}(\mathrm{s}(x), 0) &\to \mathrm{true}, \\ \mathrm{p}(0) &\to 0, & \mathrm{p}(\mathrm{s}(x)) &\to x \end{aligned}$$

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CoLoR: certification of termination proofs

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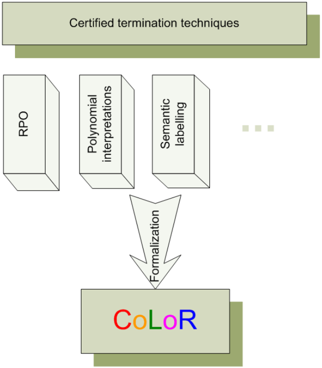
- Rainbow: a tool for translation from proofs in TPG format to Coq proofs, using results from CoLoR.

CoLoR's architecture overview



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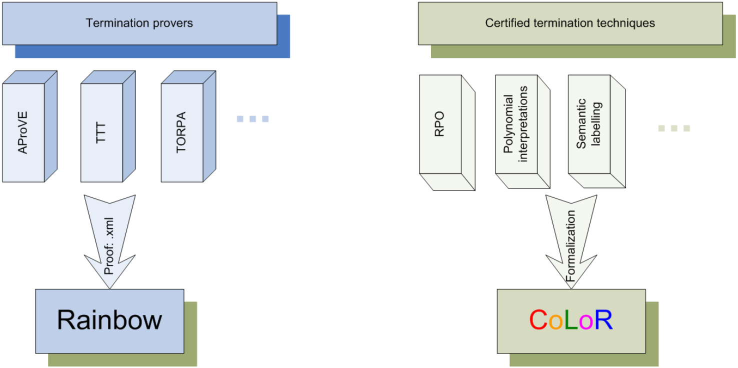

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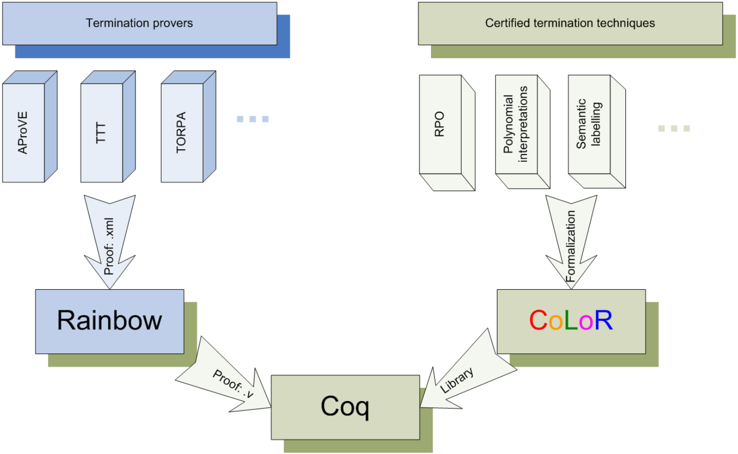

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problem set	time	s	sa	sz	saz	2007 winner
975 TRS	1 min	361	376	388	389	TPA: 354
	10 min	365	381	393	394	
517 SRS	1 min	178	312	298	320	Matchbox: 337
	10 min	185	349	323	354	

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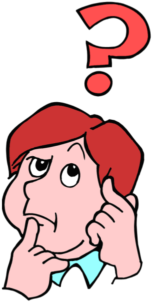
The end



Thank you for your attention.